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A ROBUST RECOVERY ALGORITHM FOR THE ROBIN INVERSE PROBLEM

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Abstract - We consider the inverse problem of identifying a Robin coefficient by performing measurements on some part of the boundary. By using a Kohn and Vogelius cost function, the inverse problem is turned into an optimisation one, which is proved to be stable with respect to the data. Robustness of the so obtained algorithm is also proved. Further investigations are related the recovery of the fast variations of the Robin coefficient, which are relevant to locate the corrosion and evaluate its level.

1 The Robin inverse problem

Corrosion of materials impacts the impedance coefficient, which intervines in a Robin boundary condition, and identifying this coefficient may thus be a way to locate the corrosion, and possibly evaluate its level, in some structure by an electric impedance tomography process. The forward problem is modelled by the Laplace equation, with a Robin boundary condition on the possibly corroded part of the boundary, whereas we hold two boundary conditions on the remaining part of the boundary : a Neumann one standing for the imposed current flux, and a Dirichlet one standing for the measured potential.

Let Ω be a connected bounded domain of $\mathbb{R}^2$. The boundary $\partial\Omega$ is assumed to be a $C^{1,\beta}$ Jordan curve, for some $\beta \in]0,1[$. Morever, let γ and Γ_N be two non empty connected open subsets of $\partial\Omega$ such that:

$$\partial\Omega = \overline{\gamma} \cup \overline{\Gamma_N}$$

The inverse problem $(\mathcal{IP})$ we are dealing with is the following:

$$(\mathcal{IP}) \begin{cases} \text{Given a prescribed flux } \phi \not\equiv 0 \text{ and measurements } f \text{ on } \Gamma_N, \\ \text{Find a function } q \text{ on } \gamma \text{ such that the solution } u \text{ of} \\ \qquad (\mathcal{NP}) \begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = \phi & \text{on} \quad \Gamma_N \\ \dfrac{\partial u}{\partial n} + qu = 0 & \text{on} \quad \gamma \end{cases} \\ \qquad \text{also satisfies } u_{|\Gamma_N} = f \end{cases}$$

We shall assume that $\phi \in L^2(\Gamma_N)$ and q belongs to a slightly restricted set of admissible parameters Q_{ad}, defined by:

$$Q_{\text{ad}} = \left\{ q \in H^1(\gamma), \text{ such that } \|q\|_{1,\gamma} \leq c \text{ and } q \geq c^{'} \chi_K \right\}$$

where c and c' are two positive constants, and K is a nonempty connected open subset of γ such that $\partial\gamma \cap K = \emptyset$.

For $q \in Q_{\text{ad}}$, we shall denote by $u^D(q,f)$ the solution of the following Robin-Dirichlet problem $(\mathcal{DP})$ using the measurements f as a Dirichlet data:

$$(\mathcal{DP}) \left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega \\ u & = & f & \text{on } \Gamma_N \\ \dfrac{\partial u}{\partial n} + q\, u & = & 0 & \text{on } \gamma \end{array} \right.$$

and by $u^N(q)$ the solution of the Neumann problem $(\mathcal{N}P)$ associated to q. The solution of the inverse problem $(\mathcal{I}P)$ will be denoted by $\bar{q}$.

$$J(q) \;=\; \int_\Omega |\nabla u^N(q) - \nabla u^D(q,f)|^2 \;+\; \int_\gamma q\, |u^N(q) \;-\; u^D(q,f)|^2$$

Referring to [4], the function J has a unique minimum which is nothing but the solution $\bar{q}$ of the inverse problem $(\mathcal{IP})$, which is thus turned to the following optimization one:

$$(\mathcal{OP}) \left\{ \begin{array}{l} \text{Find } \bar{q} \in Q_{\text{ad}} \text{ such that} \\ J(\bar{q}) \leqslant J(q) \quad \forall\, q \in Q_{\text{ad}}. \end{array} \right.$$

In order to solve the above problem using a descent method, we compute the Gâteaux-derivative of the cost function J with respect to the unknown Robin coefficient q [4].

$$\lim_{h \to 0^+} \frac{J(q + hq') - J(q)}{h} \;=\; \int_\gamma q' \,\left[(u^D(q,f))^2 \;-\; (u^N(q))^2 \right]$$

Thanks to this result, we are able to carry out a gradient algorithm in order to solve the optimization problem $(\mathcal{O}P)$. At each step of the algorithm, we need to compute the Robin-Dirichlet solution $u^D(q,f)$, and the Robin-Neumann one $u^N(q)$, no additional adjoint problem being needed in order to compute the gradient of the cost function.

Results obtained are of three kinds:

Stability of the inverse problem [3] The inverse problem $(\mathcal{I}P)$ has been proved to hold a logarithmic stability, which means that the misfit between two solutions q_1 and q_2 related to a close and smooth enough pair of data (f_1, f_2) verify :

$$\|q_1 \;-\; q_2\|_{0,\gamma} \;\leq\; C\,\varepsilon\,(\|f_1 \;-\; f_2\|_{1,\Gamma_N})$$

where C is some constant and ε a logarithmically behaving function, namely :

$$\varepsilon(x) \;=\; \frac{2 \;+\; \rho\, \log 1/x}{\rho\, \log 1/x}$$

Stability of the optimization problem [2] The optimization problem $(\mathcal{O}P)$ is proved to have solutions even though the inverse problem $(\mathcal{I}P)$ has not (case of uncompatible data), and these solutions depend continuously on the data. Given be a sequence $(f_n)_{n \in \mathbb{N}}$ of "measurements" in $H^{\frac{1}{2}}(\Gamma_N)$, we thus have:

$$\lim_{n \to \infty} \|f_n \;-\; f\|_{\frac{1}{2},M} \;=\; 0 \;\Longrightarrow\; \lim_{n \to \infty} \|q_n \;-\; \bar{q}\|_{L^2(\gamma)} \;=\; 0$$

Robustness of the algorithm The Kohn-Vogelius based algorithm turns out to be robust, which means that noisy data $f^\varepsilon \in L^\infty(\Gamma_N)$ do produce, after having been properly smoothed in a B-splines basis, impedances q^ε that converge:

$$\lim_{\varepsilon \to \infty} \|f^\varepsilon \;-\; f\|_{L^\infty(\Gamma_N)} \;=\; 0 \;\Longrightarrow\; \lim_{\varepsilon \to \infty} \|q^\varepsilon \;-\; \bar{q}\|_{L^\infty(\gamma)} \;=\; 0$$

So far, no quantitative estimates on the robustness have been obtained. The issue is under study, but numerical results show a very good behaviour of the algorithm to that respect.

Actually, its regularizing properties even turn out to be excessive, since the impedance oscillations - which provide with valuable information on the corrosion levels - are rubbed out. Several methods have therefore been tested in order to recover further information on these oscillations.

- Relaxation on a Fourier basis is preferrable to global descent methods, for these latters ignore the higher frequencies which are directions of slow energy decreases.
- Anti-dumping, which means enhancing higher frequencies, may also be of some help to that end, although it is done in an heuristic way and cannot thus be controlled.
- Finally, considering the KV function as a regularizer, rather than the cost function itself, also brings a better recovery of the impedance variations.

Still, none of these methods is totally satisfactory, for what they come up with is nothing much better than the mean value of the impedance. Capturing its oscillations needs thus to make use, in a second step of the algorithm, of an alternative method.
We now decompose the impedance to be recovered in two parts, the first one standing for its mean value, and the second for its "fast variations":

$$q = q_1 + q_2$$

Having recovered the mean value q_1 by using the above described method, we now built a new cost function using regularization on the second part q_2, in the space G introduced by Y. Meyer [6], and which has been succesfully used to recover textures in image processing since it allows oscillations:

$$G = \{q \in L^2(\gamma) \,;\, q(s) = h'(s),\, h \in L^\infty(\gamma) \,;\, h = 0 \text{ on } \partial\gamma\}$$

The space G can also be characterized [6] as follows:

$$G = \{q \in L^2(\gamma) \,;\, \int_\gamma q = 0\}$$

and norm of G is provided by:

$$\|q\|_G = \inf_{h'=q} \|h\|_{L^\infty(\gamma)}$$

The non differentiable cost function to be minimized is given by

$$J_1(q_2) = \frac{1}{2}\,\|u^N(q) - f\|^2_{L^2(\Gamma_N)} + \frac{\varepsilon}{2}\,\|q_2\|^2_G$$

Besides the difficulties related to the L^∞ norm, the dependence of u on q adds non linearity to the above functional. This is the reason why we shall transform it a little bit in order to make its handling through an optimization process easier.

1.1 Identification of the oscillatory coefficients

Let q_1 be the "mean value" of the Robin coefficient, obtained as a result of the Kohn and Vogelius algorithm. We are now going to use q_1 as a first guess in the forthcoming algorithm.
Denoting by g the quantity

$$g = -\,q\,u^N(q)$$

$u^N(q)$ also solves the following full Neumann problem:

$$(\mathcal{P}_g)\begin{cases} \Delta v(g) = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial v(g)}{\partial n} = \phi & \text{on} \quad \Gamma_N, \\ \dfrac{\partial v(g)}{\partial n} = g & \text{on} \quad \gamma, \\ \displaystyle\int_{\Gamma_N} v(g) = \int_{\Gamma_N} f, & \end{cases}$$

where the pair (ϕ, g) of Neumann boundary conditions obviously verify:

$$\int_\gamma g + \int_{\Gamma_N} \phi = 0$$

Rather than driving our minimization algorithm by q, let us drive it by g, since the dependence of $v(g)$ is affine with respect to g. Should we calculate the "right" g (which means the one such that $v(g) = u^N(q)$), we would immediately derive q from it that

$$q = -\frac{g}{v(g)}$$

Up to a constant g is an element of G. Actually, $\tilde{g}$ is given by

$$\tilde{g} = g + \frac{1}{\text{meas}(\gamma)} \int_{\Gamma_N} \phi$$

In order to find out the "right" g, one needs to minimize the discrepancy between $v(g)$ and the measurements, which can be done by a least squares method, with an appropriate additional regularization. We shall use the following:

$$J_2(\tilde{g}) = \frac{1}{2} \|v(g) - f\|^2_{L^2(\Gamma_N)} + \frac{\varepsilon}{2} \|\tilde{g}\|^2_G$$

Now, let $g_0 = -\dfrac{1}{\text{meas}(\gamma)} \displaystyle\int_{\Gamma_N} \phi$, consider $v_0 = v(g_0)$ and define

$$w(\tilde{g}) = v(g) - v_0$$

$w(\tilde{g})$ thus solves the following problem:

$$\begin{cases} \Delta w(\tilde{g}) & = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial w(\tilde{g})}{\partial n} & = 0 \quad \text{on} \quad \Gamma_N, \\ \dfrac{\partial w(\tilde{g})}{\partial n} & = \tilde{g} \quad \text{on} \quad \gamma, \\ \displaystyle\int_{\Gamma_N} w(\tilde{g}) = 0, \end{cases}$$

which makes $w(\tilde{g})$ linear with respect to $\tilde{g}$. Moreover, $w(\tilde{g})$ should fit the data f_0 in order that $v(g)$ fits the measurements f:

$$f_0 = f - v_0 |_{\Gamma_N}$$

Now, the cost function we are trying to minimize over G can also be written as

$$J_2(\tilde{g}) = \frac{1}{2} \|w(\tilde{g}) - f_0\|^2_{L^2(\Gamma_N)} + \frac{\varepsilon}{2} \|\tilde{g}\|^2_G$$

Considering that $\|\tilde{g}\|_G = \|h\|_{L^\infty(\gamma)}$ where h is the single function belonging to $H^1_0(\gamma)$ such that $\dfrac{\partial h}{\partial \tau} = \tilde{g}$, our cost function can also be written as

$$J_3(h) := J_2(\tilde{g}) = \frac{1}{2} \|w(g(h)) - f_0\|^2_{L^2(\Gamma_N)} + \frac{\varepsilon}{2} \|h\|^2_{L^\infty(\gamma)}$$

where $w^h := w(g(h))$ solves the problem:

$$(\mathcal{P}_h) \begin{cases} \Delta w^h & = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial w^h}{\partial n} & = 0 \quad \text{on} \quad \Gamma_N, \\ \dfrac{\partial w^h}{\partial n} & = \dfrac{\partial h}{\partial \tau} \quad \text{on} \quad \gamma, \\ \displaystyle\int_{\Gamma_N} w^h = 0. \end{cases}$$

The L^∞ norm in its expression makes J_3 non differentiable, which may cause troubles in its numerical handling. A similar (though finite dimensional) problem has been recently tackled by Chaabane and Kunisch [5], who solved it by pseudo dualization, which consists in finding out an easier to handle minimization problem, the dual problem which is the one we are dealing with. After properly discretizing our problem, we shall be able to use their technique.

1.2 The discrete minimization problem

In this section we shall suppose that the domain Ω is polygonal, and its boundary Γ is discretized by $(n+1)$ points $(\theta_i)_{i=0,\cdots,n+1}$ with $\theta_0 = \theta_{n+1}$. Moreover, the $(k+1)$ points $(\theta_i)_{i=0,\cdots,k}$ are situated on γ, whereas $(\theta_i)_{i=k,\dots,n+1}$ are situated on Γ_N. Points θ_0 and θ_k are then the two connecting points between the two of the parts of the boundary, Γ_N and γ.
Let us denote by $(\chi_i)_{i=0,\cdots,n}$ the piecewise linear shape functions defined by $\chi_i(\theta_j) = \delta_{ij}$ for $i,j = 0,\cdots,n$.

Since the discrete- h function we are trying to recover is such that $h|_{\partial\gamma} = 0$, it spans over the shape functions $(\chi_i)_{i=1,\cdots,k-1}$:

$$h = \sum_{i=1}^{k-1} h_i\,\chi_i$$

where $h_i = h(\theta_i)$; $i = 1,\cdots,k-1$. From the discretization of the *linear* problem $(\mathcal{P}_h)$ using piecewise linear finite elements, we derive a $(k-1)\times(n+2-k)$ matrix M such that:

$$w^h_{|\Gamma_N} = M\,h = \sum_{i=1}^{k-1} h_i\,M\chi_i = \sum_{i=1}^{k-1} h_i\,w_i$$

where $(w_i)_{i=1,\cdots,k-1}$ solve problems $(\mathcal{P}_i)$ for $i = 1,\cdots,k-1$.

$$(\mathcal{P}_i)\begin{cases} \Delta w_i = 0 & \text{in } \Omega, \\ \dfrac{\partial w_i}{\partial n} = 0 & \text{on } \Gamma_N, \\ \dfrac{\partial w_i}{\partial n} = \dfrac{\partial \chi_i}{\partial \tau} & \text{on } \gamma, \\ \displaystyle\int_{\Gamma_N} w_i = 0. \end{cases}$$

Let us define:

$$\alpha_{ij} = w_i(\theta_j)\ ;\ i = 1,\cdots,k-1 \text{ and } j = k,\cdots,n+1$$

It follows that:

$$w_i|_{\Gamma_N} = \sum_{j=k}^{n+1} \alpha_{ij}\,\chi_j$$

and consequently:

$$w^h|_{\Gamma_N} = \sum_{i=1}^{k-1}\sum_{j=k}^{n+1} (\alpha_{ij}\,h_i)\,\chi_j$$

The discrete optimization problem we are now dealing with is the following:

$$(\mathcal{OP}_d)\begin{cases} \min\left(\dfrac{1}{2}\,\|M\,h - f_0\|^2_{L^2(\mathbb{R}^{n+2-k})} + \dfrac{\varepsilon}{2}\,\|h\|^2_{L^\infty(\mathbb{R}^{k-1})}\right) \\ h \in \mathbb{R}^{k-1} \end{cases}$$

We recall that $f_0 = f - u^0 \in L^2(\Gamma_N)$, which leads to its following discretization:

$$f_0 = \sum_{i=k}^{n+1} f_0^i\,\chi_i$$

with $f_0^i = f_0(\theta_i)$, for $i = k,\cdots n+1$.

The continuous operator M is injective since $M\,h = 0 \Rightarrow \dfrac{\partial h}{\partial \tau} = 0$ on γ, and thus $h = 0$ since h vanishes at both extremal points of γ. So the matrix M will be non–singular, provided n and k are properly chosen $(n+2-k \geq k-1)$. Considered as an operator mapping $\mathbb{R}^{k-1}$ on $R(M)$, M is therefore invertible. Let M^{-1} be its inverse, which operates on $R(M)$, which is isomorphic to $\mathbb{R}^{k-1}$ since its dimension is $k-1$, and let M^{-*} be its adjoint operator. The main result of this section is the following:

Theorem 1.1 *Problem $(\mathcal{OP}_d)$ has a unique solution $h \in \mathbb{R}^{k-1}$ and it is the dual problem of the following:*

$$(\mathcal{OP}_d{-}*)\left\{\begin{array}{l}\min\left(\dfrac{1}{2}\,\|M^{-*}\,z\|^2_{L^2(\mathbb{R}^{n+2-k})} + <z, M^{-1}f_0> + \dfrac{1}{2\,\varepsilon}\|z\|^2_{L^1(\mathbb{R}^{k-1})}\right)\\ z \in \mathbb{R}^{k-1}\end{array}\right.$$

This result implies solving a total variation like problem, for which several methods have already been developed. This work is currently on the way, and we expect some numerical results to be presented during the conference.

2 Conclusions

In this contribution, we have first presented an algorithm, based on the Kohn and Vogelius cost function, which is an energy discrepancy between two boundary value problems which can be solved from the precribed flux and the measured data, in order to retrieve the Robin coefficient in the related inverse problem. The algorithm has been proved to be stable and robust with respect to noisy data. The price to pay for that robustness, is however excessive in the present case, since the algorithm rubs out oscillations of the unknown coefficient, which are helpful for the location of the corrosion, and the evaluation of its effects. This is the reason why we have been driven to study, as a second step of the algorithm, a method to retrieve the fast variations of the coefficient, once its mean value has been recovered by the KV algorithm. This is done through regularization in a space introduced by Yves Meyer for textures in image processing, which seems convenient since it all! ows oscillations. The price to pay is now to face non linearities, which is done through a pseudo dualization method proposed by Chaabane and Kunisch.

Numerical trials are currently on the way, which should allow us to present some numerical results during the conference.

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